

Statistical Equilibria of Uniformly Forced Advection Condensation

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We examine the state of statistical equilibrium attained by a uniformly forced condensable substance subjected to advection in a periodic domain. In particular, we examine the probability density function (PDF) of the condensable substance in the limit of rapid condensation. The constraints imposed by this limit are pointed out and are shown to result in a PDF — whenever the advecting velocity field admits a diffusive representation — that features a peak at small values, decays exponentially and terminates in a rapid "roll-off" near saturation. Possible physical implications of this feature as compared to a PDF which continues to decay slowly are pointed out. A set of simple numerical exercises which employ lattice maps for purposes of advection are performed to test these features. Despite the simplicity of the model, the derived PDF is seen to compare favourably with PDF's constructed from isentropic specific humidity data. Further, structure functions associated with the condensable field are seen to scale anomalously with near saturation of the scaling exponents for high moments — a feature which agrees with studies of high resolution aircraft data.

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I. INTRODUCTION

Advection-Diffusion-Condensation (ADC) is a variant of the familiar passive advection-diffusion problem [1], wherein apart from molecular diffusion a tracer is subject to an additional sink associated with the process of condensation. A study of the interplay of these processes is motivated by the need to understand the large scale distribution of water vapor in the troposphere [2],[3],[4]. Indeed, the prominent role played by water vapor in various problems related the Earth's climate [5] — for example, via the water vapor feedback in the greenhouse effect [6] — makes this an important issue. In this context, ADC serves as an idealized problem for water vapor in the troposphere [7]. The mixing ratio $q(\vec{x}, t)$ of a scalar tracer subject to advection, diffusion and condensation is governed by

$$\frac{\partial q}{\partial t} + (\vec{u} \cdot \nabla)q = \eta \nabla^2 q + S(q, q_s) + F \quad (1)$$

here \vec{u} and F are the advecting velocity field and the forcing respectively and η is the diffusivity of the condensable substance. In the present work we shall assume the domain to be spatially bi-periodic and restrict our attention to the nondiffusive limit, $\eta = 0$. The problem thus consists of the advection of a passive tracer supplied by the source F , and removed by a sink associated with condensation. This sink is represented as

$$S = -\frac{1}{\tau}[q - q_s(\vec{x})] \quad \text{if } q > q_s \\ = 0 \quad \text{if } q \leq q_s \quad (2)$$

where $q_s(\vec{x})$ is the saturation mixing ratio — a prescribed function. Hence with $\eta = 0$, the ADC problem is closely related to the interaction of smooth advection with linear damping [8]. The ADC problem also bears a similarity recent studies of "active" processes when coupled with fluid advection, such as the evolution of chemically [9] or biologically [10] active substances and the issue of phase separation in immiscible fluids [11].

Our aim is to examine the state of statistical equilibrium attained by (1) and (2) in the limit $\tau \rightarrow 0$ or that of *rapid condensation*. This limit implies that upon advection if $q(\vec{x}, t) > q_s(\vec{x})$, then the parcel's mixing ratio is instantaneously reset to the saturation mixing ratio at that location. Of course, two immediately apparent opposing long time limits of (1) and (2) are : (i) Uniform forcing with $u = 0 \Rightarrow q(\vec{x}, t) \rightarrow q_s(\vec{x})$ and (ii) A sufficiently mixing flow with $F = 0 \Rightarrow q(\vec{x}, t) \rightarrow \min_{\vec{x}}[q_s(\vec{x})]$. Hence, to achieve a non-trivial state of statistical equilibrium we require both $\vec{u}, F \neq 0$.

In the absence of diabatic effects, parcel motion in the midlatitude troposphere is restricted to two-dimensional surfaces of constant potential temperature, i.e. to isentropic surfaces [12]. Our objective is to understand the probability density function (PDF) of the water vapor mixing ratio along these midlatitude isentropic surfaces [13]. On average the thermal structure of the midlatitude troposphere is such that as one progresses polewards, the temperature along isentropic surfaces decreases in a fairly linear manner. Given the sensitivity of the saturation mixing ratio (via the Clausius-Clapeyron relation) to the temperature, following [7] we take q_s to vary exponentially with y . As we are in a periodic domain, $q_s(\vec{x}) = q_s(y) = \exp(-\alpha|y - \frac{L}{2}|)$ ($0 \leq y \leq L$) where $\alpha > 0$ and increasing α yields progressively steeper profiles which fall off symmetrically from $y = \frac{L}{2}$. This yields an idealized model problem which retains the essence of the much more complex atmospheric problem that motivates our work. Even though the real atmosphere does not conform exactly to the idealizations, progress can be made through a detailed solution of this model problem.

II. THE PDF IN THE LIMIT OF RAPID CONDENSATION

A. General formulation and boundary conditions

Before we proceed to the PDF in the general case, consider the limit alluded to in the Introduction. Specifically, $\vec{u} = 0$ with constant forcing. On one hand this case is quite straightforward but on the other it exhibits some pathologies inherent in dealing with the PDF. From (1) at a location x_0, y_0 for $t > t_1 = q_s(x_0, y_0)/F$ we have

$$q(x_0, y_0, t) = q_s(x_0, y_0) + F\tau[1 - \exp(\frac{-(t - t_1)}{\tau})] \quad (3)$$

As (3) shows, in general the domain will become supersaturated. But for rapid condensation, as claimed, $q(x_0, y_0, t) \rightarrow q_s(x_0, y_0) \forall (x_0, y_0)$. Note that the limit of saturation is approached from above. Physically this implies that as $\tau \rightarrow 0$, we simultaneously have $q(x, y, t) \rightarrow q_s(x, y)$ such that the condensation sink balances the forcing.

To estimate the PDF (denoted by $P = P(x, y, q, t)$), we decompose the stationary solution as $P(x, y, q) = P_1(x, y)F_1(q|x, y)$ where $F_1(q|x, y)$ is the conditional PDF of q conditioned on (x, y) . From the preceeding discussion, $F_1(q|x, y) = \delta[q - q_s(x, y)]$. As $\vec{u} = 0$, for an unbiased estimate it is natural to take $P_1(x, y)$ to be uniform resulting in $P(x, y, q) = \delta[q - q_s(x, y)]$. In essence we end up with a δ function supported on the fixed point of the dynamical system (1) with $\vec{u} = 0$ [14]. In this particular case, the system does not have any other invariant PDF and the δ function PDF is meaningful (as will be seen later), but in general to avoid such complications we restrict our attention to functions that are reasonably smooth.

Proceeding to the general case [26], the Liouville or transport equation satisfied by the PDF is

$$\begin{aligned} \frac{\partial P(x, y, q, t)}{\partial t} + \frac{\partial(u_i P)}{\partial x_i} + \frac{\partial[(F + S)P]}{\partial q} &= 0 \\ \text{with } \int_{\vec{x}} \int_0^\infty P(x, y, q, t) d\vec{x} dq &= 1 \quad \forall t \end{aligned} \quad (4)$$

as in general, i.e. outside the limit of rapid condensation, $0 \leq q(x, y, t) \leq \infty$ and apart from periodicity in \vec{x} we require $P(x, y, \infty, t) = 0$.

In (x, y, q) space the above can be viewed as resulting from a *probability current* $\vec{U}P$ where $\vec{U} = [u, v, (F + S)]$. An immediate consequence of rapid condensation is a restriction on q i.e. $q(x, y, t) \leq q_s(x, y)$. Till now our choice of forcing has been completely arbitrary, at this stage we restrict ourselves to a forcing which is positive definite and also satisfies $\frac{\partial F}{\partial q} = 0$. This implies, whatever the initial condition on q is, after a finite time $\min(q) = \min(q_s)$. Hence, $\min(q_s) \leq q(x, y, t) \leq q_s(x, y)$ and using (2), (4) reduces to

$$\frac{\partial P(x, y, q, t)}{\partial t} + \frac{\partial(u_i P)}{\partial x_i} + \frac{\partial(FP)}{\partial q} = 0$$

with $\int_{\vec{x}} \int_{\min(q_s)}^{q_s(x, y)} P(x, y, q, t) d\vec{x} dq = 1 \quad \forall t$ (5)

and, as noted, apart from the restriction on the forcing function, we require P to be relatively smooth. Also, note that now the domain is periodic in (x, y) and is bounded by the surfaces $q = \min(q_s)$ and $q = q_s(x, y)$. Further, periodicity and the imposition of rapid condensation imply $[\vec{U}P]|_{\partial\mathbf{D}} = 0$ and in essence (5) represents the evolution of a PDF via an incompressible flow within a bounded domain with the normalization constraint implicitly providing the required boundary condition. Interestingly, similar integral formulations for the PDF arise in studies of spiking neurons (see for example Fusi & Mattia [15] and the references therein).

B. Solutions for present saturation profile

In the present case, as the saturation mixing ratio is purely a function of y , if the forcing is also taken to be of the form $F = F(y)$ (in fact we will focus on the uniformly forced case) — it is reasonable to look for solutions which are independent of x . Of course, it is the realization of statistical equilibrium without the presence of gradient fields (as $\eta = 0$) that makes progress possible in the present case [27]. This circumvents the need to estimate conditional expectations of the scalar dissipation (or diffusion) which complicate advection-diffusion problems [16], [17], [18].

1. The uniform case

Inspecting (5) it is evident that without any further assumptions, as per the usual situation involving the evolution of a PDF via an incompressible flow, a valid stationary solution is that of a uniform distribution in (y, q) . To obtain the PDF of q from this uniform distribution we estimate the probability that $q < Q$ as a function of Q (i.e. Q represents the sample space variable corresponding to q). Utilizing the rapid condensation normalization constraint, $P(q, y)$ has to be integrated with respect to the saturation curve $q_s(y)$. Therefore

$$Pr(q < Q) = \frac{I_1 + I_2}{I_t} \tag{6}$$

Here I_1, I_2 correspond to the regions $y \leq Y$ and $y > Y$ where $Y = Z(Q)$ (Z being the inverse of $q_s(y)$). In the present case, as q_s is symmetric about $y = \frac{L}{2}$ we need only consider half of the domain ($L/2 \leq y \leq L$) with $q_s = \exp(-\alpha y)$, i.e. $Z(Q) = -\log(Q)/\alpha$. Specifically,

$$\begin{aligned} I_1 &= \int_{L/2}^{Z(Q)} \int_{Q_2}^Q P(q, y) dq dy \\ I_2 &= \int_{Z(Q)}^L \int_{Q_2}^{q_s(y)} P(q, y) dq dy \end{aligned} \tag{7}$$

where $Q_2 = \min(q_s)$. I_t in (6) is the same as I_2 but with $L/2$ as the lower limit of integration in the outer integral, which ofcourse is nothing but the normalization constraint in (5). Hence,

$$\text{PDF}(Q) = \frac{d(I_1 + I_2)}{dQ} \tag{8}$$

Substituting $P(y, q) = \text{const.}$ in (7) and (8) yields $\text{PDF}(Q) \sim Z(Q)$ i.e. $\text{PDF}(Q) \sim \log(Q^{-\frac{1}{\alpha}})$ which can be seen in the upper panel of Fig. (1). In essence, for a uniform distribution the constraint of rapid condensation forces the PDF to reflect the saturation profile.

2. The eddy-diffusion case

We now assume that the effect of the fluctuating velocity in the Liouville equation admits a diffusive representation — say by an eddy diffusivity κ_e . This would be the case if the parcel trajectories consisted of independent Brownian motion, for example. Admittedly, this is a fairly severe assumption in that mixing by multiple scale velocity fields rarely follows a simple diffusive prescription [19]. But from a tropospheric viewpoint, it is known that the meridional (y - direction) Lagrangian eddy-velocity correlation function decays quite rapidly (on the order of a few days) [20] — hence, in this context the assumption of an eddy diffusivity may not be completely unreasonable. Note that the eddy diffusivity we refer to here is an eddy diffusivity applied to probability evolution in (y, q, t) space. This is not the same as characterizing mixing by an eddy diffusivity to an evolution equation for a coarse-grained q in (y, t) space. Indeed, in [7] it is shown that the Brownian model yields different coarse-grained q statistics than the mean-field diffusivity model; it is suggested further that the Brownian motion model constitutes a minimal model for investigation of the interplay of transport processes with a nonlinear sink term such as condensation. In that sense, the Brownian model is worth of study in and of itself.

With the eddy-diffusivity (5) yields

$$\frac{\partial P}{\partial t} - \kappa_e \frac{\partial^2 P}{\partial y^2} + F(y) \frac{\partial P}{\partial q} = 0 \quad (9)$$

For uniform forcing $F(y) = \delta$ (a constant), as $P > 0$ and is periodic in y , by inspection the ansatz

$$P(q, y) \sim \exp(-k^2 q) [\cos(\lambda y) + \sin(\lambda y)] ; \text{ where } \lambda = \frac{1}{4}, k^2 = \lambda^2 \frac{\kappa_e}{\delta} \quad (10)$$

satisfies a stationary form of (9). Indeed, for a quantitative estimate we would need to fix the constants that arise in (10) via the normalization constraint. But for a qualitative estimate substituting $P(q, y)$ in (7) and setting $dG/dy = [\cos(\lambda y) + \sin(\lambda y)]$ for notational simplicity, yields

$$\text{PDF}(Q) \sim \exp(-k^2 Q) [G(Z(Q)) - G(0)] \quad (11)$$

Substituting for $G(y)$

$$\text{PDF}(Q) \sim \frac{\exp(-k^2 Q)}{\lambda} \{ \sin[\lambda Z(Q)] - \cos[\lambda Z(Q)] + 1 \} \quad (12)$$

and finally substituting for $Z(Q)$

$$\text{PDF}(Q) \sim \frac{\exp(-k^2 Q)}{\lambda} \{ \sin[\log(Q^{\frac{-\lambda}{\alpha}})] - \cos[\log(Q^{\frac{-\lambda}{\alpha}})] + 1 \} \quad (13)$$

The dependence of the PDF on $\frac{\kappa_e}{\delta}$ is shown in the lower panel of Fig. (1). In all cases, the PDF has a peak for small q decreases exponentially for intermediate values of q (the slope increases with $\frac{\kappa_e}{\delta}$) and then rolls-off as $q \rightarrow \max(q_s)$.

Comparing this with the uniform case (upper panel of Fig. (1)), it is evident that the form of the PDF is still controlled by the saturation mixing ratio profile. But the details, such as the slope of the PDF for intermediate values of Q , are dependent on the eddy-diffusivity and forcing. Also, it is worth noting that the PDF is in marked contrast to what one would obtain for a fully saturated domain. For example in the saturated case we had $P(q, y) = \delta(q - q_s(y))$, this yields $\text{PDF}(Q_{\text{sat}}) \sim -dF(Q)/dQ$ which implies a power law (Q^{-1} , with no "roll-off" for high Q) when $q_s = \exp(-\alpha y)$. At first sight the roll-off in (13) appears to be an obvious result of enforcing rapid condensation, but note that the saturated case considered above also conforms with the rapid condensation limit yet it yields a very different PDF. This is illustrated more clearly by choosing $q_s(y)$ to be a linearly decreasing function of y , now the saturated PDF is uniform, whereas (12) again yields a slowly decaying PDF with a smooth roll-off at large q .

With regards to water vapor, given the logarithmic dependence of the infrared cooling on the water vapor mixing ratio, it follows that fluctuations in the mixing ratio actually increase infrared cooling [7]. Hence a slowly decaying PDF, by increasing the probability of encountering a large fluctuation (as compared to a normal PDF), increases the efficiency of radiative cooling to space. In this regard, the rapid roll-off of the tail of the PDF could play a strong role in that it sharply decreases the possibility of very large fluctuations — for example, in comparison to a advection-diffusion model of water vapor distribution where one would encounter exponential tails under homogeneous forcing [21] — and would imply a reduction in the infrared cooling or in other words an increase in the temperature of the surface to maintain energy balance. Quantifying this effect by using a full-fledged atmospheric radiation code is a project we hope to pursue in the near future.

III. NUMERICAL SIMULATIONS

A primary assumption in our derivation was the use of an "eddy diffusivity" to represent the effect of an advecting velocity field. As mentioned, in most flows of interest the velocity field is expected to be quite coherent and might not conform to a diffusive representation. To test the stringency of this assumption we numerically simulate the ADC system by employing a lattice map for purposes of smooth large scale advective mixing [22],[23]. The velocity field is

$$\begin{aligned} u(x, y, t) &= f(t) A_1 \sin(B_1 y + p_n) \\ v(x, y, t) &= (1 - f(t)) A_2 \sin(B_2 x + q_n) \end{aligned} \quad (14)$$

where $A_1 = 0.75, A_2 = 0.5, B_1 = B_2 = 2.5$, $f(t)$ is 1 for $nT \leq t < (n+1)T/2$ and 0 for $(n+1)T/2 \leq t < (n+1)T$. $p_n, q_n \in [0, 2\pi]$ are random numbers selected at the beginning of each iteration, i.e., for each period T . Advection is implemented via sequential integer shifts in the x and y directions on a square lattice (see [22] for details). Apart from its numerical efficiency the lattice map has the advantage of preserving moments, i.e. it does not introduce spurious diffusion into the problem.

To test the features of the PDF seen in Fig. (1), we set $\delta = \min(q_s)/\Delta t$. Note that as the advective map has no time scale, we set $\Delta t = 1$ to match an iteration of the mapping. Numerically this amounts to incrementing the mixing ratio at every site on the lattice by $\min(q_s)$ at every iteration. Now as $\min(q_s)$ is a function of α , δ decreases with increasing α . Interestingly, we observe that even though the maximum attainable mixing ratio is $\max(q_s)$, the system achieves equilibrium for much smaller maxima in q field. A snapshot of the equilibrium condensable field for $\alpha = 1, 1.25$ and 1.5 is shown in Fig. (2) — note the increase in "graininess" of the field with α . The PDF's for these cases can be seen in Fig. (3) — the plotted PDF's are averages over the last couple of iterations of the map. In spite of the non-local nature of the mixing protocol the shapes of the PDF's display the features anticipated from the eddy-diffusive case, in particular we see the decrease in the slope of the PDF with δ (the advective map is unchanged so κ_e is the same in all the simulations) also the characteristic roll-off induced via the form of the saturation profile in conjunction with the limit of rapid condensation is evident.

Another measure of quantifying intermittency in a field is to examine its structure functions, defined as $S_n(|\vec{r}|) = \langle |q(\vec{x} + \vec{r}) - q(\vec{x})|^n \rangle$ (in the present situation $\langle \cdot \rangle$ denotes a spatial average) higher moments (i.e. larger n) of the structure functions are most sensitive to the "roughest" regions of the field. For the condensable field, we observe that $S_n(r) \sim r^{\zeta_n}$ for $l_1 \leq r \leq l_2$ where $l_1 \rightarrow 0$ as we are dealing with a non-diffusive problem and l_2 is an outer scale which is smaller than the size of the domain. More to the point, the exponents ζ_n are anomalous in the sense $\zeta_n < n\zeta_1$ for $n > 1$. In fact, we observe near saturation of the scaling exponents (i.e. ζ_n tends to a constant) for large n . Both the scaling of the structure functions and the extracted scaling exponents are shown in Fig. (4). The anomalous behaviour is physically anticipated as smooth advection by itself leads to the formation of sharp fronts in finite time — hence one has a field composed of smooth regions interrupted by step like discontinuities. Indeed the combination of these two structures yields a ζ_n profile which increases linearly for $n \leq 1$ and then saturates to a constant (i.e. extreme intermittency) for $n > 1$ [24]. As the additional presence of condensation does not involve any smoothing of the field, we do not expect it to be able to mollify the discontinuities created via advection. But rapid condensation provides a bound for the magnitude of the jump across the advective discontinuity, i.e. $|q(x+r) - q(x)| \leq 1 - \exp(-\alpha L/2)$, hence we expect a weak dependence of ζ_n on α . Indeed, this

qualitative reasoning is borne out in the lower panel Fig. (4). In a similar vein, it has been shown (under further assumptions regarding the nature of the velocity field) that an interplay of advection with linear damping produces severe anomalous scaling [8]. Interestingly, such anomalous behaviour — with saturation of ζ_n for large n — has been observed in an analysis of specific humidity fluctuations from high resolution aircraft data in the troposphere [25].

IV. CONCLUSIONS AND ATMOSPHERIC DATA

We have examined the state of equilibrium achieved via the interplay of smooth advection and condensation. Exploiting the attainment of equilibrium without the presence of gradient fields allows us to make progress on the equation governing the PDF of the condensable substance. In particular, the limit of rapid condensation admits a simplified Liouville equation with an integral normalization constraint on the PDF. Assuming a straightforward uniform solution to the governing equation shows the PDF to be tied to the particular saturation profile under consideration. Relaxing this uniformity and assuming an eddy-diffusive nature for the advective velocity fields shows the form of the PDF to still be controlled by the saturation profile but details, such as the rate of decay, to be sensitive to the eddy-diffusivity and strength of the forcing. Indeed, numerical simulations employing a lattice map for advective purposes reproduce these features. Further, scaling exponents extracted from structure functions of the condensable field show anomalous behaviour. Physically, the anomalous scaling is anticipated given the tendency of undiffused smooth advection to create sharp fronts. In fact, the near saturation of ζ_n for large n is a reflection of these fronts providing the leading contribution to the higher order structure functions.

As mentioned in the Introduction, our motivation is to understand the distribution of water vapor along midlatitude isentropic surfaces. With this in mind we construct PDF's of the midlatitudinal specific humidity field along the 300 K isentrope using data from the ECMWF reanalysis (ERA40) project. As is seen in Fig. (5), despite the simplifying assumptions made in our derivation, the PDF's from data can be approximated by the eddy-diffusive PDF. In fact, in Fig. (5) we've also plotted the PDF that results from assuming uniformity (for the same α) and as is seen even though the primary shape of the data PDF follows from the saturation profile in conjunction with rapid condensation, it is the enhanced slope via $\frac{\kappa}{\delta}$ that captures the decay of the PDF. Note that for large q the PDF's from data deviate significantly from the eddy-diffusive estimate. As we have considered the nondiffusive limit a possible source of this discrepancy might lie in the homogenization induced by diffusion. Secondly, as the source of water vapor lies in the tropics a boundary forcing might be more appropriate for the actual atmospheric problem. Even so, these moderately encouraging results lead us to conjecture that the ADC model — in the limit of rapid condensation with the proper saturation profile — driven by idealized velocity fields might be of use in predicting the statistical properties of the large scale distribution of water vapor in the midlatitude troposphere.

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- [26] We are grateful to the referee who pointed out problems in our original formulation of the PDF equation. Indeed, this section would be much weaker if not for his comments.
- [27] A similar absence of gradient fields in equilibrium has been indirectly exploited in the study of advection with linear damping [8].

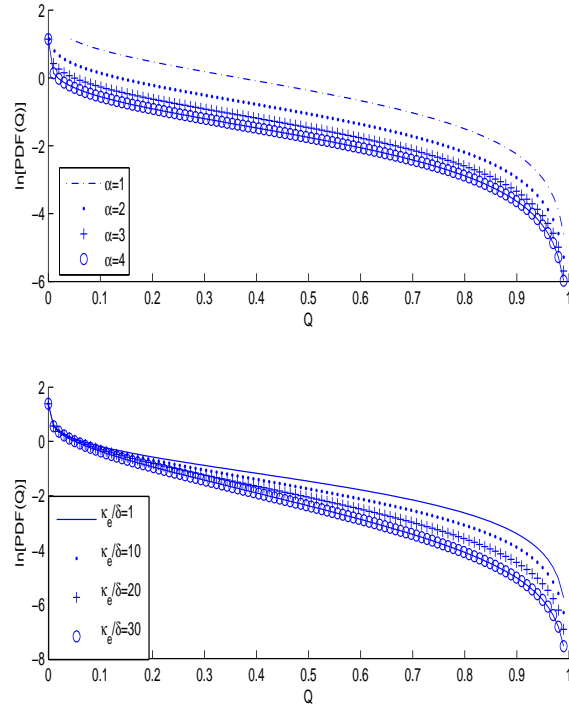


FIG. 1: Upper Panel : The PDF for the uniform case showing the dependence on α . Lower Panel : The PDF in (13) as a function of $\frac{\kappa_e}{\delta}$ with $\alpha = 3$.

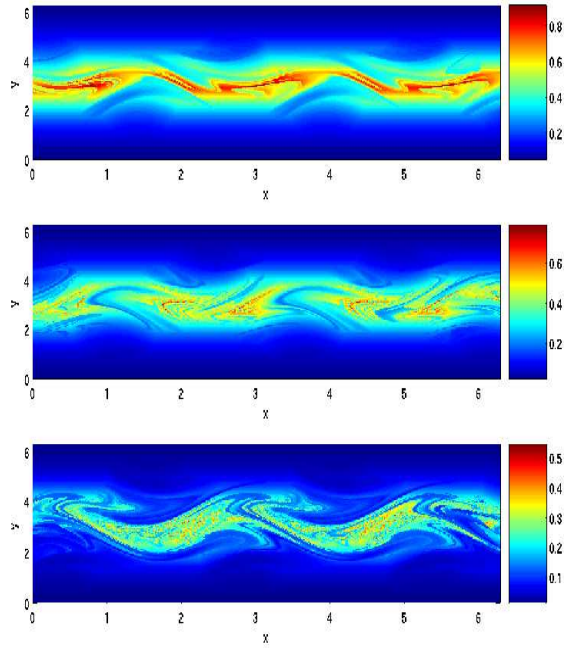


FIG. 2: A snapshot of the condensable field under the action of the map specified in (14). The snapshots are after the PDF has settled into an invariant shape. The upper, middle and lower panels correspond to $\alpha = 1, 1.25$ and 1.5 respectively with $\delta = \min(q_s)$.

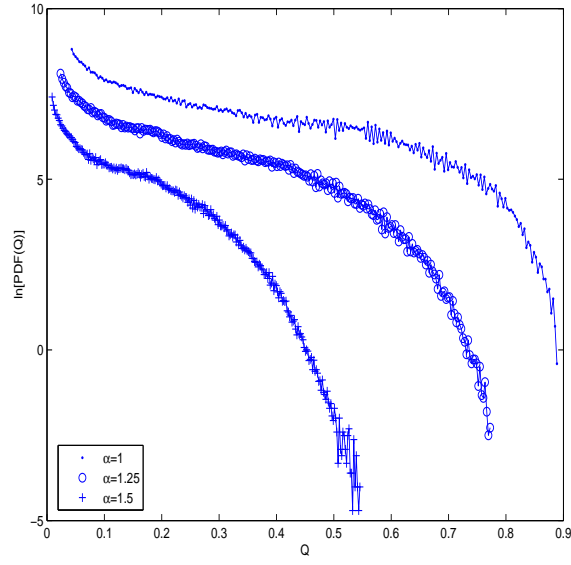


FIG. 3: The PDF's of the condensable field for $\alpha = 1, 1.25$ and 1.5 . The curves have been shifted for clarity. Note, as per Fig. (1), as α increases δ decreases resulting in an increase in the slope (as κ_e is the same in every case) of the PDF. Also notice the increase in the sharpness of the peak with α .

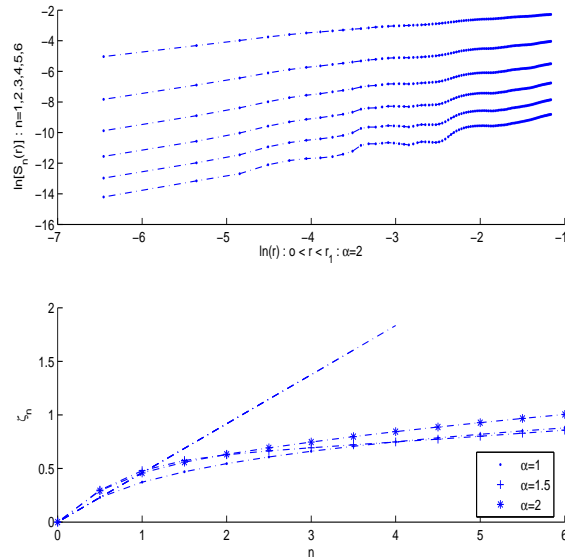


FIG. 4: Upper Panel : Plots of $\log(S_q(r))$ Vs. $\log(r)$ for the first six moments of the equilibrium condensable field with $\alpha = 2$. Lower panel : Scaling exponents extracted from above (for $\alpha = 2$) and similar plots (not shown) for $\alpha = 1, 1.5$.

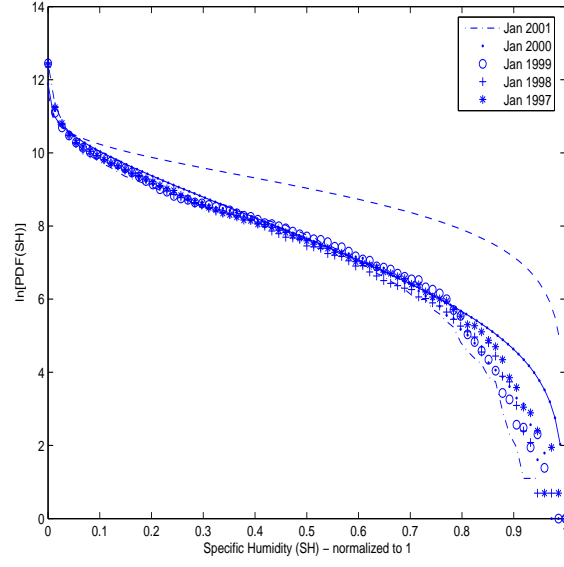


FIG. 5: The normalized specific humidity PDF's in the midlatitudes (between 30° and 60° in both hemispheres) along the 300 K isentropic surface for Jan 97-01 from ECMWF data. The solid line with dots is a plot of (13) (shifted vertically by a constant) with $\frac{\kappa_e}{\delta} = 45, \alpha = 3$ and for comparison the dash-dot line is a plot of the PDF resulting from uniformity, again with $\alpha = 3$.